Announcements

1) Math Advising Session
11:30 - 1:30 CB 2047
(Math Library) - There
will be pizza!

2) Survey on CTools -
worth 2 points extra
credit - but only if
everyone does it! Due
Sunday 4/21

Orthogonal Projections

(section 6.3)

Note: Not an orthogonal matrix!

The only orthogonal projection P that satisfies $P^tP = I_n$ is $P = I_n$!

Recall definition:

Two vectors v and w in $\mathbb{R}^n$ are said to be orthogonal if $v \cdot w = 0$.

A collection $\{v_1, \ldots, v_k\}$ of vectors is said to be orthogonal if v_i is orthogonal to v_j for all $i \neq j$.

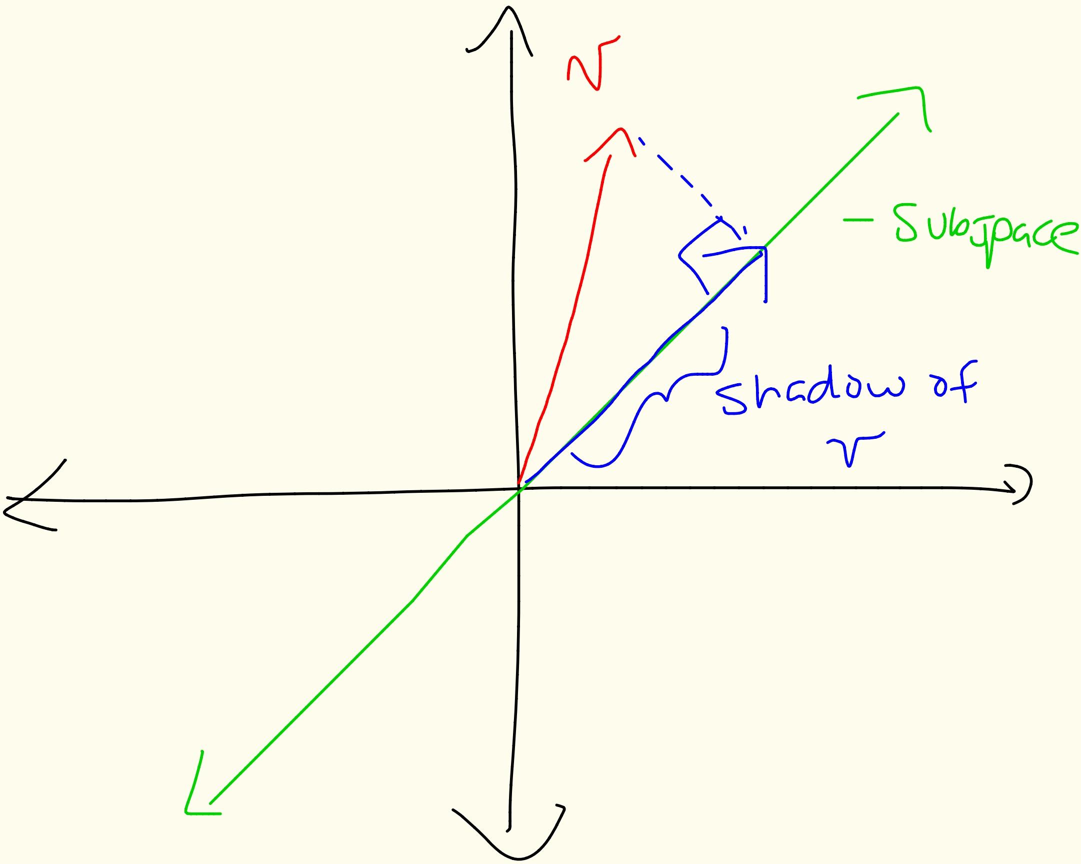
Example 1: ($\mathbb{R}^2$)

In $\mathbb{R}^2$, any subspace is either $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$, $\mathbb{R}^2$, or one-dimensional. Let's look at the last case.

All such subspaces are lines through the origin.

Take any vector not on the line. It casts a "shadow" on the subspace. This shadow is the orthogonal projection of the vector onto the subspace.

Picture:



Definition: (orthogonal projection)

Let $P: \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. We say P is an orthogonal projection if

$$\boxed{\begin{gathered} P^2 = P \\ P = P^t \end{gathered}}$$

In $M_n(\mathbb{R})$, you always get O_n and I_n as orthogonal projections.

But there are always more!

Note: (diagonalization)

Since $P = P^t$, P is diagonalizable. The only eigenvalues of P are zero and one.

Extra Credit: If $P : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal projection onto a line through the origin, find the matrix of P with respect to the standard basis.

Due Friday 4/19

Example 2: ($\mathbb{R}^4$)

Define

$T : \mathbb{R}^2 \to \mathbb{R}^4$

$$T(x,y) = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}}_{v} x + \underbrace{\begin{bmatrix} 0 \\ -6 \\ 13 \\ 4 \end{bmatrix}}_{w} y$$

T determines a 2-dimensional subspace of $\mathbb{R}^4$ via $\mathrm{Ran}(T)$.

Find a formula for the orthogonal projection onto $Ran(T)$.

Find an orthogonal basis for $Ran(T)$.

The vectors v and w are linearly independent but not orthogonal.

Let's make a vector u in $Ran(T)$ with $u \neq 0$, u orthogonal to v.

We want

$v \cdot u = 0$ and

Since u is in $\mathrm{Ran}(T)$,

there are numbers a and b

with $u = av + bw$.

Since $v \cdot (cu) = 0$ for

any constant c, we can

assume $a = 1$.

$$0 = v \cdot u$$

$$= v \cdot (v + bw)$$

$$= v \cdot v + v \cdot (bw)$$

$$= v \cdot v + b(v \cdot w)$$

Solving for b, we get

$$\boxed{b = -\frac{v \cdot v}{v \cdot w} = -\frac{\|v\|_2^2}{v \cdot w}}$$

Then $u = v - \frac{\|v\|_2^2}{v \cdot w} w$

Claim: If s is in $\mathbb{R}^4$,

we define

$$\boxed{P(s) = \frac{s \cdot u}{\|u\|_2^2} u + \frac{s \cdot v}{\|v\|_2^2} v}$$